DIFFERENTIAL MANIFOLDS HOMEWORK 2

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1. Exercise 1.20

(a). Suppose X is star shaped about x_0. Then our homotopy is naturally $F(t,x) := tx + (1-t)x_0$, $x \in X$ and $t \in [0,1]$. By definition of star shaped, this segment must lie within X.

(b). Setting $x_0 = 0$ in the above homotopy formula, we have that $F(t,x) = tx$. Now consider a p-form ω. First compute $F^*\omega$:

$$
\begin{aligned}
F^*\omega_{(s,x)}\big((\delta s, \delta x), (\delta' s, \delta' x)\big) &= \omega_{sx}(\delta(sx), \delta'(sx)) \\
&= \omega_{sx}(x\delta s + s\delta x,\ x\delta' s + s\delta' x)
\end{aligned} \tag{1.1}
$$

Where δs, δx are tangent vectors at (s,t). Now take the interior product with $V(s,x) = (1,0)$:

$$
\begin{aligned}
i_V\omega_{sx}(x\delta s + s\delta x,\ x\delta' s + s\delta' x) &= \omega_{sx}(x\delta s(1,0) + s\delta x(1,0),\ x\delta' s + s\delta' x) \\
&= \omega_{sx}(x,\ s\delta x) + \omega_{sx}(x,\ x\delta s) \\
&= \omega_{sx}(x,\ \delta x)s
\end{aligned} \tag{1.2}
$$

Now consider the pullback of $G := e^{tV} \circ J$ which takes $x \mapsto (t,x)$. Then this is merely the change of variable sending $s \mapsto t$. To see this, merely rewrite:

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$$= \omega_{sx}(x,\ \delta x)s = \omega_{F(s,t)}(\delta F(s,x)(1,0), \delta' F(s,x))$$

Then our pullback becomes:

$$[e^{tV} \circ J]^* \omega_{F(s,t)}(\delta F(s,x)(1,0), \delta' F(s,x)) = \omega_{F(G(x))}(\delta F(G(x))(1,0), \delta' F(G(x))) \tag{1.3}$$
$$= \omega_{tx}(x,\ \delta x)t$$

Finally, integrating from 0 to 1:

$$H\omega(\delta x) = \int_0^1 \omega_{tx}(x,\ \delta x) t dt$$

And for a constant form, $\omega_{tx} = \omega$ so that

$$H\omega(\delta x) = \omega(x,\ \delta x) \int_0^1 t dt = \frac{1}{2}\omega(x,\ \delta x)$$

(c). Consider the case for any p-form ω. Our pullback F^* becomes:

$$F^*\omega\Big((\delta^1 x, \delta^1 x),\ \dots,\ \delta^p s, \delta^p x)\Big) = \omega_{sx}\big(\delta^1(sx),\ \dots, \delta^p(sx)\big) \tag{1.4}$$
$$= \omega_{sx}\big(s\delta^1 x + x\delta^1 s,\ \dots, s\delta^p x + x\delta^p s\big)$$

And computing the interior product similarly:

$$i_V \omega_{sx}\big(s\delta^1 x + x\delta^1 s,\ \dots, s\delta^p x + x\delta^p s\big) = \omega_{sx}\big(s\delta^1 x(1,0) + x\delta^1 s(1,0),\ \dots, s\delta^p x + x\delta^p s\big) \tag{1.5}$$
$$= \omega_{sx}\big(x,\ \dots, s\delta^p x + x\delta^p s\big)$$
$$= \omega_{sx}\big(x, \delta^2 x,\ \dots,\ \delta^p x\big) s^{p-1}$$

Arguing identically as in part (b), the pullback by $e^{tV} \circ J$ merely sends $s \mapsto t$, so we derive the general case:

$$H\omega(\delta^2 x,\ \dots, \delta^p x) = \int_0^1 \omega_{tx}(x,\ \delta^2 x,\ \dots,\ \delta^p x)t^{p-1}dt$$

And for a constant form,

$$H\omega(\delta^2 x,\ \dots, \delta^p x) = \frac{1}{p}\omega(\delta^2 x,\ \dots, \delta^p x)$$

And we are done.

2. Problem 1.21

(a). We first check that $\theta = (2xy + z^2)\mathrm{d}x + x^2\mathrm{d}y + 2xz\mathrm{d}z$ is in fact closed:

(2.1)
$$\begin{aligned}\mathrm{d}\theta &= (2y\mathrm{d}x + 2x\mathrm{d}y + 2z\mathrm{d}z) \wedge \mathrm{d}x + (2x\mathrm{d}x) \wedge \mathrm{d}y + (2z\mathrm{d}x + 2x\mathrm{d}z) \wedge \mathrm{d}z \\ &= -2x\mathrm{d}x \wedge \mathrm{d}y + 2z\mathrm{d}z \wedge \mathrm{d}x + 2x\mathrm{d}x \wedge \mathrm{d}y - 2z\mathrm{d}z \wedge \mathrm{d}x \\ &= 0\end{aligned}$$

So that θ is closed. Now, using the formula from 1.20, we know:

$$\mathrm{d}H\omega = \omega$$

Where $H\omega = \int_0^1 \omega_{tx}(\delta x)dt$. Then, in our case we see the triple $(x, y, z) \mapsto (tx, ty, tz)$ and $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) \mapsto (x, y, z)$ (this works because evaluating our one forms merely results a replacement). Using this, θ is mapped to the 0-form:

$$\theta \mapsto t^2(2xy + z^2)x + t^2x^2y + t^2 2xz^2$$

And integrating from 0 to 1, we find our antiderivative:

$$H\theta = x^2y + xz^2$$

(b). Again, we check that $\omega = (x^2 - 2xy)\mathrm{d}y \wedge \mathrm{d}z + (y^2 - 2yz)\mathrm{d}z \wedge \mathrm{d}x + (z^2 - 2zx)\mathrm{d}x \wedge \mathrm{d}y$ is closed:

$$
\begin{aligned}
\mathrm{d}\omega &= (2x\mathrm{d}x - 2y\mathrm{d}x - 2x\mathrm{d}y) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\quad + (2y\mathrm{d}y - 2z\mathrm{d}y - 2y\mathrm{d}z) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&\quad + (2z\mathrm{d}z - 2z\mathrm{d}x - 2x\mathrm{d}z) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= (2x - 2y)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + (2y - 2z)\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + (2z - 2x)\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= (2x - 2y + 2y - 2z + 2z - 2x)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = 0
\end{aligned} \tag{2.2}
$$

So ω is exact. Now, using the derived formula from 1.20 again, $(x, y, z) \mapsto (tx, ty, tz)$ and we evaluate our vector fields at each of x, y and z. The final result becomes

$$H\omega = \frac{yz(2x + y - 3z)}{4}\mathrm{d}x - \frac{xz(3x - 2y - z)}{4}\mathrm{d}y + \frac{xy(x - 3y + 2z)}{4}\mathrm{d}z$$

And this is an antiderivative (note that this is not unique! We can find a simpler antiderivative just by inspection).

3. Exercise 1.22

Compute the pullback of $F(\rho, \phi)$ as recommended in the hint. We see that $x = \rho\cos\phi$, $y = \rho\sin\phi$, and

$$\mathrm{d}x = \cos\phi\mathrm{d}\rho - \rho\sin\phi\mathrm{d}\phi$$

$$\mathrm{d}y = \sin\phi\mathrm{d}\rho + \rho\cos\phi\mathrm{d}\phi$$

Plugging in,

(3.1)

$$F^*\omega = \rho^{-2}(\rho \sin\phi\cos\phi \mathrm{d}\rho + \rho^2\cos^2\phi\phi\mathrm{d}\phi) - \rho^{-2}(\rho\sin\phi\cos\phi\mathrm{d}\rho - \rho^2\sin^2\phi\mathrm{d}\phi$$

$$= \rho^{-2}\big(\rho^2(\cos^2\phi + \sin^2\phi)\big)\mathrm{d}\phi = \mathrm{d}\phi$$

This shows that ω is closed automatically since $\mathrm{dd}\phi = 0$.

Now, note that this would imply that $\phi = \phi(x, y)$ is our antiderivative. Hence, as $\phi \to \pi$, we see that $\phi(x, y) \to \phi(-1, 0) = \pi$, and likewise as $\phi \to -\pi$, $\phi(x, y) \to \phi(-1, 0) = -\pi$. But this is impossible, as $\phi(-1, 0)$ now takes on two different values.